ECS455: Chapter 5

OFDM

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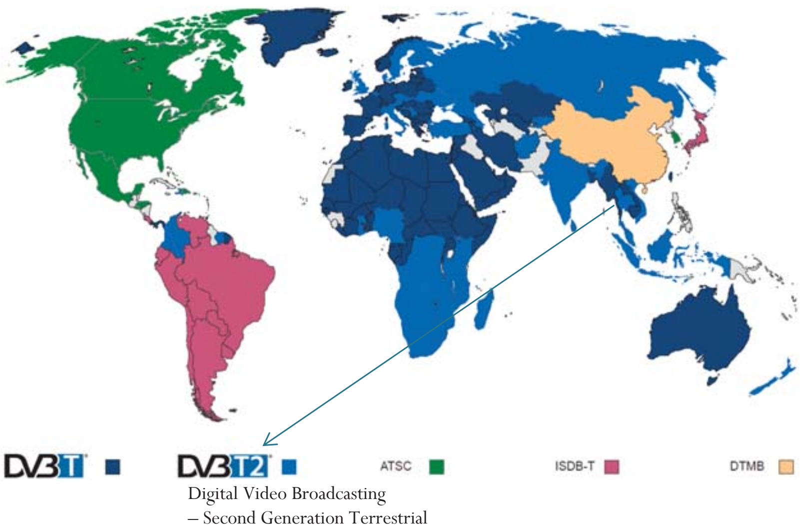
Tuesday	**14:20-15:20**
Wednesday	**14:20-15:20**
Friday	**9:15-10:15**

OFDM Applications

- 802.11 **Wi-Fi**: a/g/n/ac versions
- **DVB-T** (Digital Video Broadcasting — Terrestrial)
  - terrestrial digital TV broadcast system used in most of the world outside North America
- DMT (the standard form of **ADSL** - Asymmetric Digital Subscriber Line)
- **WiMAX, LTE (OFDMA)**

Wireless	Wireline
IEEE 802.11a, g, n (WiFi) Wireless LANs	ADSL and VDSL broadband access via POTS copper wiring
IEEE 802.15.3a Ultra Wideband (UWB) Wireless PAN	MoCA (Multi-media over Coax Alliance) home networking
IEEE 802.16d, e (WiMAX), WiBro, and HiperMAN Wireless MANs	PLC (Power Line Communication)
IEEE 802.20 Mobile Broadband Wireless Access (MBWA)	
DVB (Digital Video Broadcast) terrestrial TV systems: DVB-T, DVB-H, T-DMB, and ISDB-T	
DAB (Digital Audio Broadcast) systems: EUREKA 147, Digital Radio Mondiale, HD Radio, T-DMB, and ISDB-TSB	
Flash-OFDM cellular systems	
3GPP UMTS & 3GPP@ LTE (Long-Term Evolution) and 4G	

Side Note: Digital TV

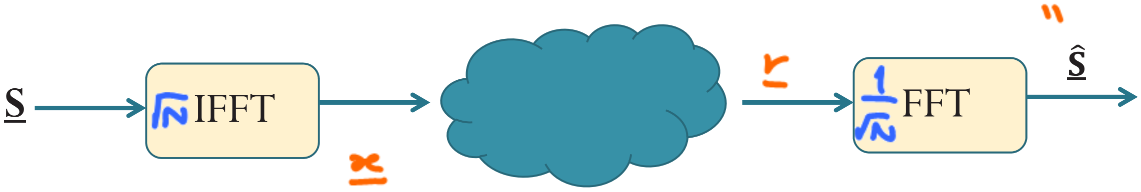


Japan: Starting July 24, 2011, the analog broadcast has ceased and only digital broadcast is available.
US: Since June 12, 2009, full-power television stations nationwide have been broadcasting exclusively in a digital format.

OFDM: Overview

- Let $\underline{\mathbf{S}} = (S_1, S_2, \ldots, S_N)$ contains the information symbols.

$\underline{x}C^{-1}$
$=$

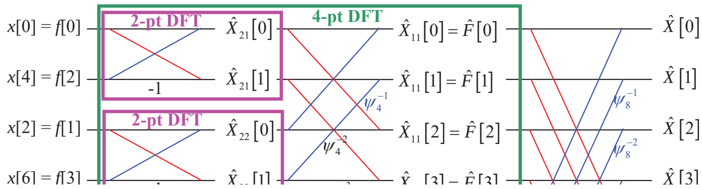


Inverse fast Fourier transform

Fast Fourier transform

$\underline{x} = \underline{S}C$

multiplication by a matrix C whose rows are orthogonal



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OFDM

5.1 Implementation: DFT and FFT

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Review: DS-CDMA

The BS receives $\underline{\mathbf{r}} = \underline{\mathbf{s}}\mathbf{C}$.
The message from the k^{th} user can be recovered via $\hat{s}_k = \frac{1}{N}\underline{\mathbf{r}} \cdot \underline{\mathbf{c}}^{(k)}$.
Alternatively, can recover all messages simultaneously: $\hat{\underline{\mathbf{s}}} = \frac{1}{N}\underline{\mathbf{r}}\mathbf{C}^T$.

BS wants to transmit s_k to the k^{th} user.
(s_1 to the 1st user, s_2 to the 2nd user, …)
BS transmits $\underline{\mathbf{x}} = \sum_k s_k \underline{\mathbf{c}}^{(k)} = \underline{\mathbf{s}}\mathbf{C}$

The transmitted signals $s_k \underline{\mathbf{c}}^{(k)}$ are combined in the air
$\underline{\mathbf{x}} = \sum_k s_k \underline{\mathbf{c}}^{(k)} = \underline{\mathbf{s}}\mathbf{C}$.

The k^{th} user (MS) wants to send s_k.
The k^{th} user (MS) transmits $s_k \underline{\mathbf{c}}^{(k)}$.

Each user (MS) receives $\underline{\mathbf{r}} = \underline{\mathbf{s}}\mathbf{C}$.
The k^{th} user (MS) can recover its message from $\hat{s}_k = \frac{1}{N}\underline{\mathbf{r}} \cdot \underline{\mathbf{c}}^{(k)}$

OFDM and CDMA

- CDMA's key equation $\underline{\mathbf{s}} = \frac{1}{N}(\underline{\mathbf{s}}\mathbf{C})\mathbf{C}^T$
  - All the rows of $\mathbf{C}$ are orthogonal
- Key property of $\mathbf{C}$:

$$\mathbf{C}\mathbf{C}^T = N\mathbf{I}. \quad \Rightarrow C^{-1} = \frac{1}{N}C^T$$

- For sync. CDMA, we use the **Hadamard matrix $\mathbf{H}_N$**.
- For OFDM, we use **DFT matrix $\boldsymbol{\Psi}_N$**. "psi"
  - The matrix is complex-valued.

Discrete Fourier Transform (DFT)

Here, we work with N-point signals (finite-length sequences (vectors) of length N) in both time and frequency domain.

N numbers N numbers

$$\vec{\mathbf{x}} = \begin{pmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{pmatrix} \longrightarrow \overset{(N\text{-pt})}{\boxed{\text{DFT}}} \longrightarrow \vec{\mathbf{X}} = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{pmatrix}$$

$$\vec{\mathbf{X}} = \text{DFT}\{\vec{\mathbf{x}}\} = \boldsymbol{\Psi}_N \vec{\mathbf{x}}$$

Ψ

DFT matrix $\mathbf{\Psi}_N$

$$\mathbf{\Psi}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \psi_N^{-1} & \psi_N^{-2} & \cdots & \psi_N^{-(N-1)} \\ 1 & \psi_N^{-2} & \psi_N^{-4} & \cdots & \psi_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \psi_N^{-(N-1)} & \psi_N^{-2(N-1)} & \cdots & \psi_N^{-(N-1)(N-1)} \end{bmatrix}$$

ψ

Element on the pth row and qth column is given by

$$\boxed{\psi_N^{-(p-1)(q-1)} \text{ where } \psi_N = e^{j\frac{2\pi}{N}}}$$

Note that the "-1" are there because we start from row 1 and column 1 (not from row 0 and column 0).

$= \frac{1}{N}\Psi_N^H$

Key Property: $\boxed{\mathbf{\Psi}_N^{-1} = \frac{1}{N}\mathbf{\Psi}_N^*}$ $\Rightarrow$ $\frac{1}{\sqrt{N}}\mathbf{\Psi}_N$ is a unitary matrix

in time domain

$$\frac{1}{N}\mathbf{\Psi}_N^*\vec{\mathbf{X}} = (\mathbf{\Psi}_N)^{-1}\vec{\mathbf{X}} = \mathrm{IDFT}\{\vec{\mathbf{X}}\} = \vec{\mathbf{x}} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \vec{\mathbf{X}} = \mathrm{DFT}\{\vec{\mathbf{x}}\} = \mathbf{\Psi}_N\vec{\mathbf{x}}$$

in freq. domain

If A is symmetric,

suppose $\vec{y} = A\vec{x}$.

Then $\underline{y} = \underline{x}A^T = \underline{x}A$

Conclusion: Similar formulas for row vectors.

Example: $N = 2$

- $\psi_2 \equiv e^{j\frac{2\pi}{N}} = e^{j\frac{2\pi}{2}} = e^{j\pi} = -1$
- $\mathbf{\Psi}_2 \equiv \begin{bmatrix} 1 & 1 \\ 1 & \psi_2^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & (-1)^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H_2$
- Suppose $\vec{\mathbf{x}} = \binom{2}{5} \to \boxed{\text{DFT}} \to \vec{\mathbf{X}} = \Psi_2\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$$= \begin{bmatrix} 2+5 \\ 2-5 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

"Inverse"

$$\vec{x} = \Psi_N^{-1}\vec{X} = \frac{1}{N}\Psi_N^*\vec{X} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

```
>> fft([2 5])
ans =
     7    -3
>> ifft([7 -3])
ans =
     2     5
```

Connection to CDMA

- The rows of $\mathbf{\Psi}_N$ are orthogonal. So are the columns.
- Proof: Let $\underline{\mathbf{r}}^{(k)}$ be the k^{th} row of $\mathbf{\Psi}_N$.

$$\left\langle \underline{\mathbf{r}}^{(k_1)}, \underline{\mathbf{r}}^{(k_2)} \right\rangle = \sum_{q=1}^{N} \psi_N^{-(k_1-1)(q-1)} \left(\psi_N^{-(k_2-1)(q-1)} \right)^* = \sum_{q=1}^{N} \psi_N^{-(k_1-1)(q-1)} \psi_N^{(k_2-1)(q-1)}$$

$$= \sum_{q=1}^{N} \left(\psi_N^{(k_2-k_1)} \right)^{(q-1)} = \sum_{q=0}^{N-1} \left(\psi_N^{(k_2-k_1)} \right)^{q}$$

$$= \begin{cases} \dfrac{1-\psi_N^{(k_2-k_1)N}}{1-\psi_N^{(k_2-k_1)}} = \dfrac{1-\left(e^{j\frac{2\pi}{N}}\right)^{(k_2-k_1)N}}{1-\psi_N^{(k_2-k_1)}} = \dfrac{1-1}{1-\psi_N^{(k_2-k_1)}} = 0, & k_1 \neq k_2, \\ \displaystyle\sum_{q=0}^{N-1} (1)^q = N, & k_1 = k_2. \end{cases}$$

- So, $\mathbf{\Psi}_N$ "replaces" the role of $\mathbf{H}_N$ in CDMA.

Discrete Fourier Transform (DFT)

Matrix form:

$$\boxed{\frac{1}{N}\mathbf{\Psi}_N^* \vec{\mathbf{X}} = \text{IDFT}\left\{\vec{\mathbf{X}}\right\} = \vec{\mathbf{x}} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \vec{\mathbf{X}} = \text{DFT}\left\{\vec{\mathbf{x}}\right\} = \mathbf{\Psi}_N \vec{\mathbf{x}}}$$

Pointwise form:

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]\psi_N^{nk} = \underset{0\le n<N}{x[n]} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]\psi_N^{-nk}$$

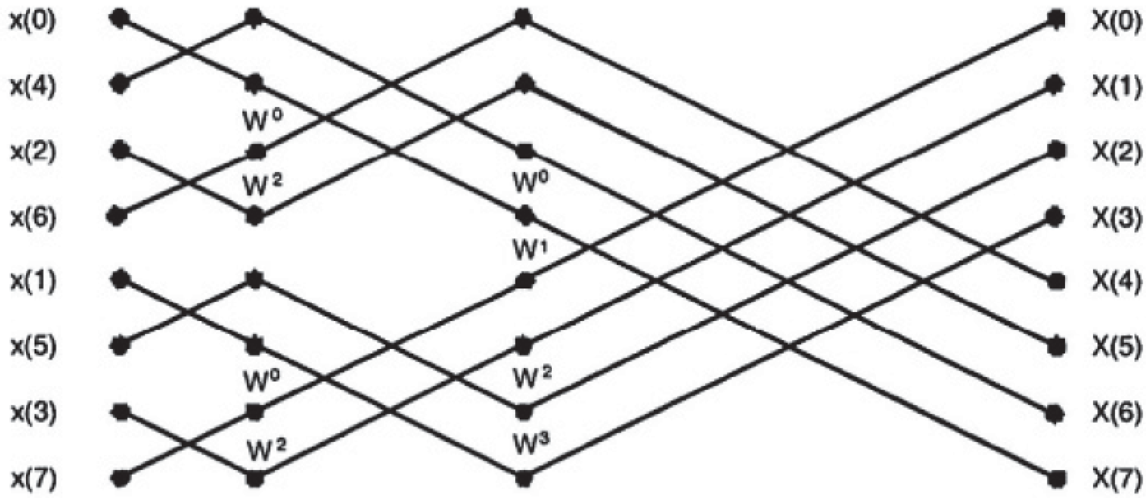
or, equivalently,

$$\boxed{\frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{jnk\frac{2\pi}{N}} = \underset{0\le n<N}{x[n]} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} \underset{0\le k<N}{X[k]} = \sum_{n=0}^{N-1} x[n]e^{-jnk\frac{2\pi}{N}}}$$

Comparison with Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} x(f) = \int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt$$

Efficient Implementation: (I)FFT

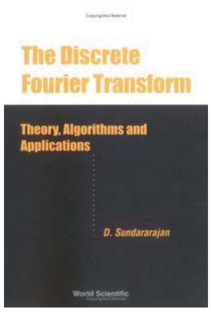


[Bahai, 2002, Fig. 2.9]

An N-point FFT requires only on the order of $N\log N$ multiplications, rather than N^2 as in a straightforward computation.

FFT

- The history of the FFT is complicated.
- As with many discoveries and inventions, it arrived before the (computer) world was ready for it.
- Usually done with N a power of two.
  - Very efficient in terms of computing time
  - Ideally suited to the binary arithmetic of digital computers.
  - Ex: From the implementation point of view it is better to have, for example, a FFT size of 1024 even if only 600 outputs are used than try to have another length for FFT between 600 and 1024.



References: E. Oran Brigham, *The Fast Fourier Transform*, Prentice-Hall, 1974.

OFDM with Memoryless Channel

$$h(t) = \beta\delta(t)$$

$\left[\text{should be } h(t) = \beta\delta(t-\tau)\right]$

$$r(t) = h(t) * s(t) + w(t) = \beta s(t) + w(t)$$

Additive white Gaussian noise

Sample every T_s/N

$$r[n] = \beta s[n] + w[n]$$

FFT

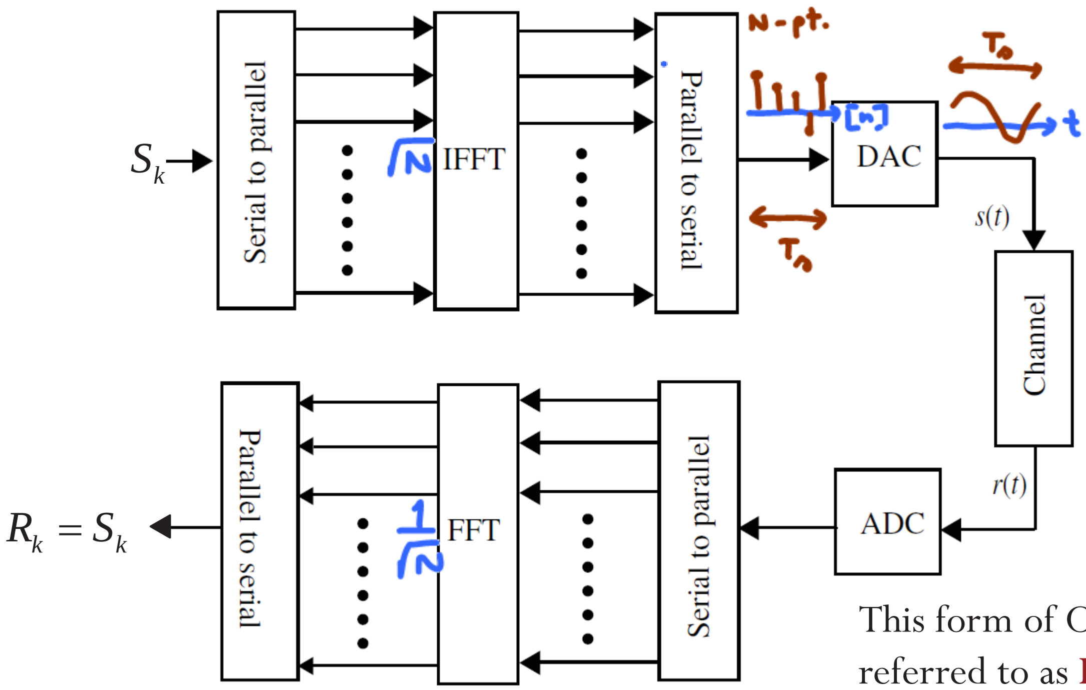
$$s[n] = \sqrt{N}\,\mathrm{IFFT}\{S\}[n]$$

$$R_k = \frac{1}{\sqrt{N}}\mathrm{FFT}\{\mathbf{r}\}[n] = \beta S_k + \frac{1}{\sqrt{N}}W_k$$

Sub-channel are independent.

(No ICI)

OFDM implementation by IFFT/FFT



This form of OFDM is often referred to as **Discrete Multi-Tone (DMT)**.